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1992 J. Phys. A: Math. Gen. 25 L755

(http://iopscience.iop.org/0305-4470/25/13/001)

J. Phys. A: Math. Gen. 25 (1992) L755–L762. Printed in the UK

LETTER TO THE EDITOR

Algebraic areas distributions for two-dimensional Levy flights

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Received 13 February 1992

Abstract. We study curves generated by planar Levy flights and calculate the probability distribution of the algebraic area 'between the curve and the chord'. We compare our analytical result with computer simulations. We also consider closed curves and give the result for the distribution of areas enclosed by those curves.

In the 1940s, P Levy solved, among others, two important problems concerning two-dimensional Brownian curves [1]. For such curves of length t, he calculated:

(i) the probability distribution $P_c(A, t)$ for a closed curve to enclose a given algebraic area A.

(ii) The probability distribution $P(A, t)$ of the algebraic area A between an open curve and the chord linking its ends.

Since that time, the study of such algebraic areas has aroused great interest among mathematicians and also physicists. For instance, the problem of the distribution $P_c(A, t)$ has been recently re-examined in various contexts [2]. In particular, transport properties in disordered materials in the presence of magnetic fields are closely related to this distribution. It allows the calculation of corrections to weak localization (anomalous magnetoresistance [3]) as well as to localization lengths in Anderson insulators [4].

Our purpose in this letter is to extend the calculation of distributions $P(A, t)$ and $P_c(A, t)$ to two-dimensional Levy flights. Standard techniques (such as path integrals) used for the treatment of Brownian motion will not be efficient. Therefore, we will have to tackle the problem in another way, essentially by solving an integro-differential equation.

First, we recall the definition of two-dimensional isotropic Levy flights. The probability for a particle starting at $\boldsymbol{r}_0(t=0)$ to reach $\boldsymbol{r}$ at time t reads [5]

$$P_\mu(\boldsymbol{r}_0, \boldsymbol{r}; t) \equiv P_\mu(\boldsymbol{r}-\boldsymbol{r}_0; t) = \left(\frac{1}{2\pi}\right)^2 \int \mathrm{d}^2\boldsymbol{k}\, \exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}_0) - t|\boldsymbol{k}|^\mu] \tag{1}$$

where $|\boldsymbol{k}| = (k_x^2 + k_y^2)^{1/2}$ and $0 < \mu \leqslant 2$.

Due to isotropy, P_μ depends only on the end-to-end distance $|\boldsymbol{r}-\boldsymbol{r}_0|$ that scales as $t^{1/\mu}$. Distribution (1) is stable and indefinitely divisible. It satisfies the Chapman–Kolmogorov equation:

$$P_\mu(\boldsymbol{r}; t) = \int \mathrm{d}^2\boldsymbol{r}'\, P_\mu(\boldsymbol{r}-\boldsymbol{r}'; t-\tau) P_\mu(\boldsymbol{r}'; \tau) \qquad 0 \leqslant \tau \leqslant t. \tag{2}$$

This property can be used to write formally $P_\mu(\boldsymbol{r}; t)$ as a path integral. Dividing $[0, t]$

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into N equal steps and taking the limit $N\to\infty$, we get, with standard notation

$$P_\mu(\boldsymbol{r};\,t)=N'\int_{\boldsymbol{r}(0)=0}^{\boldsymbol{r}(t)=\boldsymbol{r}}\mathscr{D}\boldsymbol{r}(\tau)\int\mathscr{D}\boldsymbol{k}(\tau)\exp\left(\int_0^t(\mathrm{i}\boldsymbol{k}\cdot\dot{\boldsymbol{r}}-|\boldsymbol{k}|^\mu)\,\mathrm{d}\tau\right)\tag{3}$$

(N' is a normalization factor).

For $\mu=2$, after the functional integration over $\boldsymbol{k}$ we are left with

$$P_2(\boldsymbol{r};\,t)=N'\int_{\boldsymbol{r}(0)=0}^{\boldsymbol{r}(t)=\boldsymbol{r}}\mathscr{D}\boldsymbol{r}(\tau)\exp\left(-\frac{1}{4}\int_0^t\dot{\boldsymbol{r}}^2\,\mathrm{d}\tau\right).\tag{4}$$

However, when $\mu<2$, the functional integration cannot be done and (3) remains as a formal expression that is not easily tractable. Therefore in the following, we will not pursue these lines.

Notice that $P_\mu(\boldsymbol{r};\,t)$ can be evaluated in closed form only in a few special cases:

(i) $\mu=2$ (Brownian random walks)

$$P_2(\boldsymbol{r};\,t)=\frac{1}{4\pi t}\,\mathrm{e}^{-r^2/4t}\tag{5}$$

satisfying the diffusion equation $\Delta P_2=\partial P_2/\partial t$

(ii) $\mu=1$ (Cauchy flights)

$$\begin{aligned}P_1(\boldsymbol{r};\,t)&=\frac{1}{2\pi}\int_0^\infty\mathrm{d}k\,k\,J_0(|\boldsymbol{k}|\cdot|\boldsymbol{r}'|)\,\mathrm{e}^{-t|\boldsymbol{k}|}\\&=\frac{1}{2\pi}\,\frac{t}{(t^2+r^2)^{3/2}}\end{aligned}\tag{6}$$

with $\Delta P_1=-\partial^2P_1/\partial t^2$.

More generally, for $0<\mu<2$ and $r\to\infty$, we have the asymptotic behaviour

$$\begin{aligned}P_\mu(\boldsymbol{r};\,t)&\sim\frac{C_\mu t}{r^{2+\mu}}\\C_\mu&=\frac{2^\mu}{\pi^2}\left[\Gamma\left(\frac{\mu}{2}+1\right)\right]^2\sin\left(\frac{\pi\mu}{2}\right).\end{aligned}\tag{7}$$

Equations (7) show that, for $\alpha>\mu$, the moments $\langle r^\alpha\rangle$ are infinite: P_μ is a broad distribution. It does not satisfy a diffusion-type equation. However, using (2) and (7), we can construct an integro-differential equation involving $\partial P_\mu/\partial t$:

$$\frac{\partial P_\mu(\boldsymbol{r};\,t)}{\partial t}=C_\mu\int\mathrm{d}^2\boldsymbol{r}'\left[\frac{P_\mu(\boldsymbol{r}';\,t)-P_\mu(\boldsymbol{r};\,t)}{|\boldsymbol{r}-\boldsymbol{r}'|^{2+\mu}}\right]\qquad 0<\mu<2.\tag{8}$$

Now, we consider the above-mentioned area distributions. A particle starting at M_0 ($\boldsymbol{r}_0$, $t=0$) will successively reach $M_1, M_2, \ldots$ and, finally, M_N ($\boldsymbol{r}_N$, $N\Delta t$) after a series of N Levy flights. (From M_i to M_{i+1}, the particle flies along a straight line.) A is the algebraic area enclosed by the polygon $OM_0M_1M_2\ldots M_NO$. We study the quantity

$$\begin{aligned}P(\boldsymbol{r}_0,\boldsymbol{r}_N,A,N\Delta t)&\\&=\int\mathrm{d}^2\boldsymbol{r}_1\ldots\mathrm{d}^2\boldsymbol{r}_{N-1}\,P_\mu(\boldsymbol{r}_1-\boldsymbol{r}_0;\,\Delta t)\ldots P_\mu(\boldsymbol{r}_N-\boldsymbol{r}_{N-1};\,\Delta t)\\&\quad\times\delta\left(A-\tfrac{1}{2}\boldsymbol{n}\cdot\left(\sum_{i=0}^{N-1}\boldsymbol{r}_i\times\boldsymbol{r}_{i+1}\right)\right)\end{aligned}\tag{9}$$

($\boldsymbol{n}$ is the unit vector along the positive z axis. We define $\boldsymbol{B}=B\boldsymbol{n}$.)

The probability distribution $P(A, N\Delta t)$ (A: algebraic area between the curve and the cord) will be obtained by setting $\boldsymbol{r}_0 = \boldsymbol{0}$ in (9) and integrating over $\boldsymbol{r}_n$; $P_c(A, N\Delta t)$ (A: algebraic area enclosed by a closed curve) is obtained by setting $\boldsymbol{r}_0 = \boldsymbol{r}_N = \boldsymbol{0}$ and correctly normalizing (translation invariance is used).

Expression (9) and the identity $2\pi\delta(x) = \int_{-\infty}^{+\infty} \mathrm{d}B\, \mathrm{e}^{\mathrm{i}Bx}$ lead to

$$P(\boldsymbol{r}_0, \boldsymbol{r}_N, A, N\Delta t) = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{+\infty} \mathrm{d}B\, \mathrm{e}^{\mathrm{i}BA} K(\boldsymbol{r}_0, \boldsymbol{r}_N, B, N\Delta t)$$

with

$$\begin{aligned} K(\boldsymbol{r}_0, \boldsymbol{r}_N, B, N\Delta t) \\ &= \int \mathrm{d}^2\boldsymbol{r}_1 \ldots \mathrm{d}^2\boldsymbol{r}_{N-1}\, P_\mu(\boldsymbol{r}_1 - \boldsymbol{r}_0; \Delta t) \ldots P_\mu(\boldsymbol{r}_N - \boldsymbol{r}_{N-1}; \Delta t) \\ &\quad \times \exp\left[-\frac{\mathrm{i}\boldsymbol{B}}{2}\left(\sum_{i=0}^{N-1} \boldsymbol{r}_i \times \boldsymbol{r}_{i+1}\right)\right] \end{aligned} \tag{10}$$

$(K(\boldsymbol{r}_0, \boldsymbol{r}_N, B, 0) = \delta^{(2)}(\boldsymbol{r}_N - \boldsymbol{r}_0))$.

Now, it is straightforward to deduce $((N-1)\Delta t \equiv t)$

$$\begin{aligned} K(\boldsymbol{r}_0, \boldsymbol{r}_N, B, t+\Delta t) \\ &= \int \mathrm{d}^2\boldsymbol{r}_{N-1}\, K(\boldsymbol{r}_0, \boldsymbol{r}_{N-1}, B, t) P_\mu(\boldsymbol{r}_N - \boldsymbol{r}_{N-1}; \Delta t) \\ &\quad \times \exp\left[-\mathrm{i}\frac{\boldsymbol{B}}{2}\cdot \boldsymbol{r}_{N-1} \times \boldsymbol{r}_N\right]. \end{aligned} \tag{11}$$

We consider the limit $\Delta t \to 0$. Using (1) and expanding to first order in Δt, we get

$$\begin{aligned} -\frac{\partial K}{\partial t}(\boldsymbol{r}_0, \boldsymbol{r}, B, t) \\ &= \left(\frac{1}{2\pi}\right)^2 \int \mathrm{d}^2\boldsymbol{r}'\, \mathrm{d}^2\boldsymbol{k}\, K(\boldsymbol{r}_0, \boldsymbol{r}', B, t) \\ &\quad \times \exp[-\mathrm{i}\boldsymbol{B}/2 \cdot \boldsymbol{r}' \times \boldsymbol{r}] \exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r} - \boldsymbol{r}')](|\boldsymbol{k}|^\mu). \end{aligned} \tag{12}$$

This is the integro-differential equation we have to solve.

Equation (12) can be rewritten

$$-\frac{\partial K}{\partial t}(\boldsymbol{r}_0, \boldsymbol{r}, B, t) = OK(\boldsymbol{r}_0, \boldsymbol{r}, B, t) \tag{13}$$

with

$$O = \left(\frac{1}{2\pi}\right)^2 \int \mathrm{d}^2\boldsymbol{u}\, \mathrm{d}^2\boldsymbol{k}\, (|\boldsymbol{k}|^\mu) \exp[-\mathrm{i}\boldsymbol{u}\cdot\boldsymbol{k} + (\mathrm{i}/2)\boldsymbol{u}\cdot\boldsymbol{B}\times\boldsymbol{r}]\, \mathrm{e}^{\boldsymbol{u}\cdot\nabla_r}.$$

When $\mu = 2$ (Brownian curves), the quantity $(\exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r} - \boldsymbol{r}')]|\boldsymbol{k}|^\mu)$ becomes $(-\nabla^2_{r'} \exp(\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r} - \boldsymbol{r}')))$. Integrations by parts in (12) lead to:

$$\begin{aligned} -\frac{\partial K}{\partial t}(\boldsymbol{r}_0, \boldsymbol{r}, B, t) &= (-\mathrm{i}\nabla - \tfrac{1}{2}\boldsymbol{r}\times\boldsymbol{B})^2 K(\boldsymbol{r}_0, \boldsymbol{r}, B, t) \\ &\equiv HK(\boldsymbol{r}_0, \boldsymbol{r}, B, t). \end{aligned} \tag{14}$$

Thus, for Brownian curves, the operator O (13) is simply the Hamiltonian H describing the motion of a particle of charge (-1) in a constant magnetic field $\boldsymbol{B}$. The kernel K is expressed in terms of the eigenstates of H:

$$K(\boldsymbol{r}_0, \boldsymbol{r}, B, t) = \sum_{M,p} \Psi^*_{M,p}(\boldsymbol{r}_0)\Psi_{M,p}(\boldsymbol{r}')\,\mathrm{e}^{-tE_{M,p}}$$

$$\Psi_{M,p}(\boldsymbol{r}') = \frac{1}{\sqrt{A_{M,p}}}\,\mathrm{e}^{\mathrm{i}M\theta}\exp[-|B|r^2/4]r^{|M|}L_p^{|M|}(\tfrac{1}{2}|B|r^2) \tag{15}$$

$$E_{M,p} = (2p+1+|M|)|B| + MB$$

$$A_{M,p} = \pi\left(\frac{2}{|B|}\right)^{|M|+1}\frac{(p+|M|)!}{p!}$$

(M and p integers, $p \geqslant 0$; $L_p^{|M|}$ are Laguerre polynomials). Thereafter, we will only be interested in zero angular momentum eigenstates.

Introducing $K(B, t) \equiv \int \mathrm{d}^2\boldsymbol{r}\,K(\boldsymbol{0}, \boldsymbol{r}, B, t) = (1/\cosh(Bt))$ and Fourier transforming, we get

$$P\left(A' = \frac{A}{t}\right) = \frac{1}{2\cosh(\pi A'/2)} \tag{16}$$

(A: area between the curve and the chord).

Likewise, for the areas A enclosed by closed curves

$$K_c(B, t) = K(\boldsymbol{0}, \boldsymbol{0}, B, t)/K(\boldsymbol{0}, \boldsymbol{0}, 0, t)$$

$$= \frac{Bt}{\sinh(Bt)} \qquad \left(K(\boldsymbol{0}, \boldsymbol{0}, 0, t) \equiv P_2(\boldsymbol{0}; t) = \frac{1}{4\pi t}\right) \tag{17}$$

$$P_c\left(A' = \frac{A}{t}\right) = \frac{\pi}{4}\,\frac{1}{\cosh^2(\pi A'/2)}. \tag{18}$$

Equations (16) and (18) are precisely the results obtained by Levy. However, the situation is not so simple when $\mu < 2$.

Coming back to the operator O (13), we consider $B > 0$ for the moment. We notice that the commutator $[\boldsymbol{u}\cdot\boldsymbol{\nabla}_r, \boldsymbol{u}\cdot\boldsymbol{B}\times\boldsymbol{r}']$ vanishes and introduce the right and left annihilation and creation operators [6] a_R, a_R^+, a_L, a_L^+

$$a_R = \frac{1}{2}\left(\sqrt{\frac{B}{2}}(x - \mathrm{i}y) + \mathrm{i}\sqrt{\frac{2}{B}}(P_x - \mathrm{i}P_y)\right), \ldots \tag{19}$$

$$[a_R, a_R^+] = 1$$

$$(\boldsymbol{P} = -\mathrm{i}\boldsymbol{\nabla}).$$

Using commutation relations, O can be written

$$O = \left(\frac{1}{2\pi}\right)^2 \int \mathrm{d}^2\boldsymbol{k}\,\mathrm{d}^2\boldsymbol{u}\,|\boldsymbol{k}|^\mu \exp\left[-\mathrm{i}\boldsymbol{u}\cdot\boldsymbol{k} - \frac{B}{4}u^2\right]\exp\left[\mathrm{i}\sqrt{\frac{B}{2}}(u_y + \mathrm{i}u_x)a_R^+\right]$$

$$\times\exp\left[\mathrm{i}\sqrt{\frac{B}{2}}(u_y - \mathrm{i}u_x)a_R\right]. \tag{20}$$

Performing the angular integration over $\boldsymbol{k}$ and using the notation $u_x \pm \mathrm{i} u_y = |\boldsymbol{u}|\,\mathrm{e}^{\pm \mathrm{i}\varphi}$, we develop the exponentials containing a_R^+ and a_R. After the angular integration over $\boldsymbol{u}$, we are left with the expression:

$$O = \int_0^\infty \int_0^\infty \mathrm{d}k\, \mathrm{d}u\, k^{\mu+1} \cdot u \cdot J_0(k \cdot u)\, \mathrm{e}^{-Bu^2/4} \left(\sum_{n=0}^{\infty} \frac{(-Bu^2/2)^n}{(n\,!)^2} (a_R^+)^n (a_R)^n \right). \tag{21}$$

It is, now, easy to express the operator O only in terms of the Hamiltonian H (14):

$$H = (2a_R^+ a_R + 1)B. \tag{22}$$

Finally, we get (for $B>0$ and $B<0$):

$$\begin{aligned} O &= |B|^{\mu/2} \left[\Gamma\left(\frac{\mu}{2}+1\right) \right] \left(\sum_{n=0}^{\infty} \frac{(-\mu/2)_n (-X)_n 2^n}{(n!)^2} \right) \\ (\alpha)_n &= \alpha(\alpha+1)\dots(\alpha+n-1) \\ X &= \frac{H}{2|B|} - \frac{1}{2} \end{aligned} \tag{23}$$

($\mu = 2$ leads to $O = H$).

O and H have the same eigenfunctions $\Psi_{M,p}$ (15).

The kernel K is given by:

$$K(\boldsymbol{r}_0, \boldsymbol{r}, B, t) = \sum_{M,p} \Psi^*_{M,p}(\boldsymbol{r}_0) \Psi_{M,p}(\boldsymbol{r}')\, \mathrm{e}^{-tE^{(\mu)}_{M,p}} \tag{24}$$

and the eigenvalues $E^{(\mu)}_{M,p}$ by (23) and (15). In particular [7]

$$\begin{aligned} E^{(\mu)}_{0,p} &= |B|^{\mu/2} C^{(\mu)}_p \\ C^{(\mu)}_p &= \frac{\Gamma((\mu/2)+1+p)}{p!} 2^{\mu/2} F(-\mu/2, -\mu/2, -p-\mu/2, 1/2) \end{aligned} \tag{25}$$

(F is the hypergeometric function).

Asymptotic expansion of Γ and F functions show that for large p values

$$E^{(\mu)}_{0,p} \simeq |B|^{\mu/2} (2p+1)^{\mu/2}. \tag{26}$$

(We checked numerically that this asymptotic value is rapidly attained.)

For the area A between the curve and the chord, we get the distribution:

$$P(A, t) = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{+\infty} \mathrm{d}B\, \mathrm{e}^{\mathrm{i}BA} K(B, t) \tag{27}$$

with

$$K(B, t) \equiv \int \mathrm{d}^2\boldsymbol{r}\, K(0, \boldsymbol{r}, B, t) = 2 \sum_{p=0}^{\infty} (-1)^p\, \mathrm{e}^{-tE^{(\mu)}_{0,p}}. \tag{28}$$

Introducing the scaling variable $A' = A/t^{2/\mu}$, (27) is rewritten

$$P(A') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}B\, \mathrm{e}^{\mathrm{i}BA'} K(B, 1). \tag{29}$$

We briefly study the tail of this distribution. When $|A'|\to\infty$, because of the large fluctuations of the phase factor $e^{iBA'}$, only small B values will give significant contributions to $P(A')$. Then, it is enough to take (26) for $E_{0,p}^{(\mu)}$. In those conditions, it is possible to show that

$$K(B,1)\simeq e^{-a_\mu|B|^{\mu/2}} \tag{30}$$

(a_μ is a constant) and

$$P(A')\propto|A'|^{-(1+(\mu/2))} \qquad \text{when } |A'|\to\infty \tag{31}$$

(compare with (7)).

Now, we discuss Cauchy flights ($\mu=1$) and first consider a curve consisting of only two steps ($N=2$, $\Delta t=1$, $t\equiv N\Delta t=2$; $A'=A/t^2=A/4$).

Calculation for $P(A')$ leads to:

$$\begin{aligned} P(A') &= \beta\left(\frac{\tan^{-1}(\alpha)}{\alpha}-1\right) && \text{if } |A'|<\tfrac{1}{8}\\ &= \beta\left(\frac{1}{2\alpha}L_n\left(\frac{1+\alpha}{1-\alpha}\right)-1\right) && \text{if } |A'|>\tfrac{1}{8}\\ &= \frac{8}{3\pi} && \text{if } |A'|=\tfrac{1}{8} \end{aligned} \tag{32}$$

where

$$\alpha=\left(\left|1-\frac{1}{64A'^2}\right|\right)^{1/2} \qquad \beta=\frac{8}{\pi(64A'^2-1)}$$

and

$$P(A')\propto\frac{L_n|A'|}{A'^2} \qquad \text{when } |A'|\to\infty.$$

Unfortunately, we have not been able to derive $P(A')$ in closed form for a number of steps, N, greater than 2.

So, we go to $P(A')$ with t continuous. The energy levels (25) $E_{0,p}^{(1)}$ are given by:

$$\begin{aligned} E_{0,p}^{(1)} &= |B|^{1/2}\left[\frac{2^p(p+\frac{1}{2})}{p!\sqrt{\pi}}\left(\Gamma\left(\frac{1+p}{2}\right)\right)^2\right] && p \text{ even}\\ &= |B|^{1/2}\left[\frac{2^{p+1}}{p!\sqrt{\pi}}\left(\Gamma\left(1+\frac{p}{2}\right)\right)^2\right] && p \text{ odd}. \end{aligned} \tag{33}$$

The asymptotic value, $\sqrt{(2p+1)|B|}$, is practically reached for $p\geq 4$. To understand this fact, it is interesting to consider the following integro-differential equation involving $\partial^2 K/\partial t^2$.

Using the definition

$$\frac{\partial^2 K}{\partial t^2}(\boldsymbol{r}_0,\boldsymbol{r},B,t)=\lim_{\Delta t\to 0}\frac{1}{(\Delta t)^2}[K(\boldsymbol{r}_0,\boldsymbol{r},B,t+2\Delta t)+K(\boldsymbol{r}_0,\boldsymbol{r},B,t)-2K(\boldsymbol{r}_0,\boldsymbol{r},B,t+\Delta t)]$$

with

$$\begin{aligned} K(\boldsymbol{r}_0,\boldsymbol{r},B,t+2\Delta t) &= \int d^2\boldsymbol{r}'\,d^2\boldsymbol{r}''\,K(\boldsymbol{r}_0,\boldsymbol{r}',B,t)P_1(\boldsymbol{r}''-\boldsymbol{r}';\Delta t)P_1(\boldsymbol{r}-\boldsymbol{r}'';\Delta t)\\ &\quad\times\exp[-i\boldsymbol{B}/2\cdot(\boldsymbol{r}'\times\boldsymbol{r}''+\boldsymbol{r}''\times\boldsymbol{r}')] \end{aligned} \tag{34}$$

after some algebra, we obtain

$$\frac{\partial^2 K(\boldsymbol{r}_0, \boldsymbol{r}, B, t)}{\partial t^2}$$

$$= HK(\boldsymbol{r}_0, \boldsymbol{r}, B, t) - \frac{1}{2}\int \frac{\mathrm{d}^2\boldsymbol{r}'\,\mathrm{d}^2\boldsymbol{k}}{(2\pi)^2} K(\boldsymbol{r}_0, \boldsymbol{r}', B, t)$$

$$\times \exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}') - \mathrm{i}(\boldsymbol{B}/2)\cdot\boldsymbol{r}'\times\boldsymbol{r}]$$

$$\times (|\boldsymbol{k}-(\boldsymbol{B}/4)\times(\boldsymbol{r}-\boldsymbol{r}')| - |\boldsymbol{k}+(\boldsymbol{B}/4)\times(\boldsymbol{r}-\boldsymbol{r}')|)^2 \qquad (35)$$

(H given by (14)).

Neglecting the integral term in (35) we get for the energy levels the approximate value

$$E'^{(1)}_{0,p} = \sqrt{(2p+1)|\boldsymbol{B}|} \qquad (36)$$

that is just the asymptotic value.

The integral term seems to be essentially related to non-local effects that are drastically reduced in the presence of a 'magnetic field' (this one imposes a length scale $\sim|\boldsymbol{B}|^{-1/2}$).

In figure 1, a comparison is made between (29) with $\mu = 1$ (full curve) and computer simulations of Cauchy flights (closed circles). We have generated 10 000 open curves, the number of steps for each curve being $t = 800$. The probability distribution $P(A')$ is a broad one: it decreases, when $|A'|$ is large, as $|A'|^{-3/2}$.

The agreement in figure 1 shows that the continuous limit is already attained for 800 steps. (In fact, we got the same agreement for 400 steps. Significant discrepancies only appear when the number of steps is very small.) It is worth noticing that the exact energy levels (33) (or (25)) must be used in the theoretical calculation. If we replace (33) (or (25)) by (36), the agreement is destroyed (especially for small $|A'|$ values).

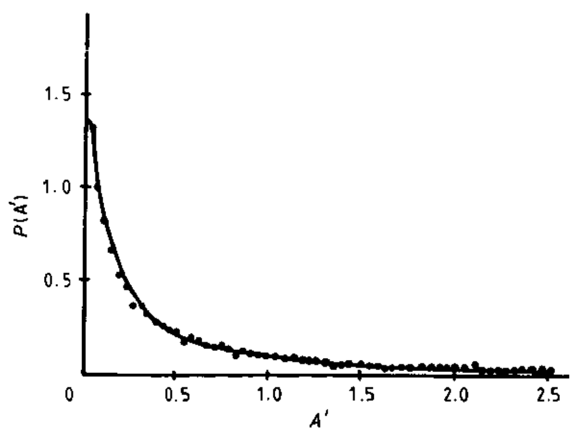

Figure 1. Computer simulations of Cauchy flights ($\mu = 1$). The probability distribution $P(A')$ is plotted as a function of the scaling variable $A' = A/t^{2/\mu}$ (here, $A' = A/t^2$). t is the number of steps (closed circles, $t = 800$) and A the algebraic area between the curve and the chord ($P(A') = P(-A')$). The full curve represents equation (29) with $\mu = 1$. $P(A')$ is a broad distribution: the tail decreases as $|A'|^{-(1+\mu/2)}$ (here, $|A'|^{-3/2}$).

To conclude, we give the probability distribution $P_c(A, t)$ for a closed curve to enclose a given algebraic area A:

$$P_c(A, t) = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{+\infty} \mathrm{d}B \, \mathrm{e}^{\mathrm{i}BA} \frac{K(\mathbf{0}, \mathbf{0}, B, t)}{K(\mathbf{0}, \mathbf{0}, 0, t)}$$
$$K(\mathbf{0}, \mathbf{0}, 0, t) \equiv P_\mu(\mathbf{0}, t) = \frac{\Gamma((2/\mu)+1)}{4\pi t^{2/\mu}}. \tag{37}$$

Of course, the scaling variable $A' = A/t^{2/\mu}$ again appears. Finally, we get

$$P_C(A') = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{+\infty} \mathrm{d}B \, \mathrm{e}^{\mathrm{i}BA'} K_C(B, 1)$$
$$K_C(B, 1) = \frac{2|B|}{\Gamma((2/\mu)+1)} \left(\sum_{p=0}^{\infty} \exp[-|B|^{\mu/2} C_p^{(\mu)}] \right) \tag{38}$$

($C_p^{(\mu)}$ given by (25)).

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